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MIXING AND SWEEPING OUT

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ABSTRACT
A weakly mixing transformation T and a sequence (d_n) are constructed such that T is uniformly mixing on (d_n), T is uniformly sweeping out on $([\alpha d_n])$ for all $\alpha \in (0, 1)$, and for all rational $\alpha \in (0, 1) T$ is not mixing on $([\alpha d_n])$.

0. Introduction

Let T be an invertible measure preserving transformation on the unit interval with Lebesgue measure and let s be an increasing sequence of positive integers. We say *T sweeps out on s* if for any set of positive measure, the union of the iterates on s has measure one. Various mixing conditions for T can be characterized by the sequences on which T sweeps out (see §1 for definitions). For example, Césaro mixing is equivalent to sweeping out on cofinite sequences; weakly mixing is equivalent to sweeping out on all sequences of positive density; mildly mixing is equivalent to sweeping out on all IP-sequences; and lightly mixing is equivalent to sweeping out on all sequences.

A transformation is *uniformly sweeping out* if for each set A of positive measure and $\varepsilon > 0$ there exists $N = N(A, \varepsilon)$ such that the union of any N iterates of A has measure exceeding $1 - \varepsilon$. In [5] it was shown that mixing implies uniform sweeping out. The converse is an open problem. Our main purpose is to provide a partial negative answer by constructing a transformation that is uniformly sweeping out on a sequence but is not mixing on the sequence. A transformation is uniformly sweeping out on a sequence s if the $N = N(A, \varepsilon)$ iterates as above are restricted to s.

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In general, every weakly mixing transformation admits eventually independent sequences [4] (see §1) and it is shown below that these sequences are uniformly sweeping out. Thus every weakly mixing transformation admits uniformly sweeping out sequences. Uniform sweeping out is also characterized by a mixing-like condition in §2.

In general, a transformation T is weakly mixing if and only if T is mixing on some sequence which then implies mixing on a sequence of density one. However, for the typical examples of weakly mixing transformations that are not mixing ([1], [9]), mixing sequences are not easily found. The stacking method for constructing the transformation in §3 is a natural construction for obtaining an explicit sequence $s = (d_n)$ on which T is mixing and uniformly sweeping out. It is then shown that T is uniformly sweeping out on $\alpha s = ([\alpha d_n])$ for all $\alpha \in (0, 1)$. It is also shown that for all rational $\alpha \in (0, 1) T$ is not mixing on αs.

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1. Preliminaries

Let $(X. \mathscr{B}, \mu)$ be the unit interval with Lebesgue measure and let T be a one-to-one measure preserving transformation mapping X into X. In addition to mixing (M), weakly mixing (WM), and Césaro-mixing (CM), we will also consider mildly mixing (MM) [8] and lightly mixing (LM) defined as follows:

(WM) $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |\mu(T^iA \cap B) - \mu(A)\mu(B)| = 0, \qquad A, B \in \mathscr{B}.$$

(CM) $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mu(T^iA \cap B) = \mu(A)\mu(B), \qquad A, B \in \mathscr{B}.$$

(MM) $$\liminf_{n\to\infty} \mu(T^nA \cap A^c) > 0, \qquad 0 < \mu(A) < 1.$$

(LM) $$\liminf_{n\to\infty} \mu(T^nA \cap B) > 0, \qquad \mu(A)\mu(B) > 0.$$

The definition of (MM) above is equivalent to the original definition [8] of no rigid factor. A transformation T is *rigid* if there exists (k_n) such that $\lim_{n\to\infty} \mu(T^{k_n}A \cap A) = \mu(A)$, $A \in \mathscr{B}$.

Chacon's transformation T that is weakly mixing but not mixing [1] is prime and has trivial commutant [2]. Therefore T is mildly mixing for otherwise T would be rigid which implies uncountable commutant [10]. However, T is not lightly mixing. Let h_n be the height of the nth column in the construction of T. If A is an interval in the kth column, $k < n$, and $B \subset (A \cup T^{-1}A)^c$, then $\mu(T^{h_n}A \cap B) = 0$. Since $h_n \to \infty$, it follows that T is not lightly mixing.

A transformation T is *mixing on* $s = (k_n)$ if $\lim_{n\to\infty} \mu(T^{k_n}A \cap B) = \mu(A)\mu(B)$, $A, B \in \mathscr{B}$. A transformation T is *uniformly sweeping out on* s if for each set A of positive measure and $\varepsilon > 0$ there exists $N = N(A, \varepsilon)$ such that $j_i \in s$, $1 \leqq i \leqq N$, implies $\mu(\bigcup_{i=1}^{N} T^{j_i}A) > 1 - \varepsilon$.

A sequence $s = (k_n)$ is an *eventually independent sequence* (e.i.s.) for T if for each finite partition P and $\varepsilon > 0$ there exists a partition Q with the same number of sets as P and a positive integer $N = N(P, \varepsilon)$ such that $|P - Q| < \varepsilon$ and $(T^{k_n}Q : n \geqq N)$ is an independent sequence [4].

An IP-set of positive integers consists of a sequence of positive integers p_i, $i \geqq 1$, together with all finite sums $p_{i_1} + p_{i_2} + \cdots + p_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. The p_i need not be distinct. An IP^*-set of positive integers is a set of positive integers that intersects every IP-set of positive integers. These definitions are due to Furstenberg [7]. Since a set of positive integers has a natural ordering, we can consider the corresponding increasing sequence of positive integers.

2. Mixing and sweeping out

We will first collect certain basic equivalences relating mixing conditions and sweeping out.

THEOREM (2.1). *The following are equivalent for a transformation.*
(a) *(CM) and sweeping out on all cofinite sequences.*
(b) *(WM) and sweeping out on all sequences of positive density.*
(c) *(MM) and sweeping out on all IP-sequences.*
(d) *(LM) and sweeping out on all sequences.*

PROOF. Ergodicity is equivalent to (CM). Since sweeping out on $s = (1, 2, 3, \ldots)$ is equivalent to ergodicity, (a) follows. Since (WM) is equivalent to mixing on a sequence of density one, (WM) implies sweeping out on all sequences of positive density. If T is not (WM), then T admits a rotation factor [3]. It follows that there exists a sequence of positive density on which T does not sweep out. Since (MM) is equivalent to IP^*-mixing ([7], p. 191), (MM) implies sweeping out on all IP-sequences. Here we use the fact that every IP-sequence contains IP-sequences beginning with arbitrarily large integers, and therefore, every IP^*-sequence meets every IP-sequence in an infinite set. If T is not (MM), then T admits a rigid factor. It follows that there exists an IP-sequence on which T does not sweep out. If T is (LM), then T sweeps out on all sequences. Conversely, suppose there exist A and B of positive measure such that $\lim_{n\to\infty}\inf \mu(T^nA \cap B) = 0$. We can choose k_n such that $\mu(T^{k_n}A \cap B) < \mu(B)/3^n$, $n \geqq 1$. Therefore T does not sweep out on (k_n).

LEMMA 2.2. *If s in an eventually independent sequence for T, then s is a uniformly sweeping out sequence for T.*

PROOF. Let $0 < \mu(A) < 1$, $P = (A, A^c)$, and $\varepsilon > 0$. Let $\alpha = \mu(A)/2$, $\beta = 1 - \alpha$, and choose M so large that $\beta^M < \varepsilon/2$. There exists $\eta > 0$ such that $\mu(A \triangle B) < \eta$ implies

$$\mu\left(\bigcup_{i=1}^{M} T^{j_i}A \triangle \bigcup_{i=1}^{M} T^{j_i}B\right) < \varepsilon/2 \tag{1}$$

for any j_i, $1 \leqq i \leqq M$. We can also choose η so small that $\mu(A \triangle B) < \eta$ implies $\mu(B) > \alpha$. Since $s = (k_n)$ is an e.i.s. for T, there exist $Q = (B, B^c)$ and $N^* = N(P, \eta)$ such that $(T^{k_n}Q : n \geqq N^*)$ is an independent sequence. In particular, if $n_i \geqq N^*$ and $j_i = k_{n_i}$, $1 \leqq i \leqq M$, then independence implies

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{M} T^{j_i}B\right) &= 1 - (1 - \mu(B))^M \\ &> 1 - \beta^M > 1 - \varepsilon/2.\end{aligned} \tag{2}$$

By (1) and (2) we can take $N = N(A, \varepsilon) = N^* + M$ for uniform sweeping out on s.

The following result is proved in [4].

THEOREM 2.3. *A transformation is weakly mixing if and only if the transformation admits an eventually independent sequence.*

COROLLARY 2.4. *A transformation is weakly mixing if and only if the transformation admits a uniformly sweeping out sequence.*

PROOF. If T is not weakly mixing, then T admits a rotation factor [3]. It follows that T cannot admit a uniformly sweeping out sequence. The converse follows by Lemma (2.2) and Theorem (2.3).

We also note that if T admits a rotation factor, then T cannot admit a mixing sequence. Thus T is weakly mixing if and only if T admits some mixing sequence. In general, a mixing sequence is not a uniformly sweeping out sequence. If T is weakly mixing but not mildly mixing then the method in [5, Example (4.6)] can be used to construct a sequence on which T is mixing but not uniformly sweeping out. A construction for a rigid weakly mixing transformation is briefly described in [6, p. 135].

An alternate notion of mixing on a sequence s is that for all A, $B \in \mathscr{B}$, and $k, j \in s$,

$$\lim_{|k-j| \to \infty} \mu(T^k A \cap T^j B) = \mu(A)\mu(B),$$

which we refer to as *uniformly mixing* [5]. Equivalently, T is uniformly mixing on a sequence if T is mixing on the sequence of differences, $s - s$. A transformation cannot be uniformly mixing on a sequence s of positive density without being mixing, since in this case, $s - s$ has bounded gaps, so that a finite union of translates of $s - s$ covers Z. However, every mixing sequence admits a uniformly mixing subsequence. Thus a transformation is weakly mixing if and only if it admits a uniformly mixing sequence. The method [5, Lemma (5.3)] can be used to show that a uniformly mixing sequence is a uniformly sweeping out sequence. In the next section we construct a transformation with uniformly sweeping out sequences that are not mixing sequences.

Uniform sweeping out can also be characterized by a mixing-like condition as follows. This condition is analogous to the mixing condition in Corollary (5.7) [5].

THEOREM 2.6. *T is uniformly sweeping out if and only if for each set A of positive measure and $\varepsilon > 0$ there exists $M = M(A, \varepsilon)$ such that $\mu(B) \geqq \varepsilon$ implies there exists a set B^* of less than M positive integers and $\mu(T^k A \cap B) \geqq (1 - \varepsilon)\mu(B)/M$, $k \notin B^*$.*

PROOF. Assume T is uniformly sweeping out and let $M = N(A, \varepsilon^2)$; hence any union of N iterates of A has measure exceeding $1 - \varepsilon^2$. Suppose there exist k_i with $\mu(T^{k_i} A \cap B) < (1 - \varepsilon)\mu(B)/M$, $1 \leqq i \leqq M$. Therefore

$$\mu\left(\left(\bigcup_{i=1}^{M} T^{k_i} A\right)^C\right) > \mu(B) - M(1 - \varepsilon)\mu(B)/M = \varepsilon\mu(B) \geqq \varepsilon^2,$$

which is a contradiction.

Conversely, let $M = M(A, \varepsilon)$ as above and let $\alpha = (1 - \varepsilon)/M$. Choose a positive integer P such that $(1-\alpha)^P < \varepsilon$. Let $N = (M+1)P$ and consider $k_1 < k_2 < \cdots < k_N$. Let $A_1 = T^{k_1}A$. If $\mu(A_1) \leqq 1 - \varepsilon$, let $B_1 = A_1^c$; hence $\mu(B_1) \geqq \varepsilon$. Let $j_1 = 1$. There exists a smallest positive integer j_2, $2 \leqq j_2 \leqq M+2$, such that $A_2 = T^{k_{j_2}}A \cap B_1$ satisfies $\mu(A_2) \geqq \alpha\mu(B_1)$. If $\mu(A_1) + \mu(A_2) \leqq 1 - \varepsilon$, let $B_2 = (A_1 \cup A_2)^c$. We also have $\mu(B_i) \leqq (1-\alpha)^i$, $i = 1, 2$. Proceeding inductively, we obtain increasing $j_i \leqq iM + i$, $1 \leqq i \leqq L$, $A_i \subset (\bigcup_{u=1}^{i-1} A_u)^c = B_{i-1}$, $A_i = T^{k_{j_i}}A \cap B_{i-1}$, $\mu(A_i) \geqq \alpha\mu(B_{i-1})$, and $\mu(B_i) \leqq (1-\alpha)^i$, $1 \leqq i \leqq L$. Since $(1-\alpha)^P < \varepsilon$, there exists $L \leqq P$ such that $\mu(\bigcup_{i=1}^L A_i) > 1 - \varepsilon$, hence $\mu(\bigcup_{i=1}^N T^{k_i}A) > 1 - \varepsilon$.

The proof of Theorem (2.1)(d) actually implies a weaker equivalent condition for lightly mixing as in Theorem (2.7) below. There is also a corresponding weaker equivalent condition for uniformly sweeping out.

THEOREM (2.7). (a) *T is lightly mixing if and only if for each pair of sets A and B of positive measure there exists* $N = N(A, B)$ *such that* $\mu(T^iA \cap B) > 0$, $i \geqq N$. (b) *T is uniformly sweeping out if and only if for each set A of positive measure and* $\varepsilon > 0$ *there exists* $M = M(A, \varepsilon)$ *such that* $\mu(B) \geqq \varepsilon$ *implies there exists a set* B^* *of less than M positive integers and* $\mu(T^iA \cap B) > 0$, $i \notin B^*$.

PROOF. For (a), note that in the proof of Theorem (2.1)(d) we have $T^{k_n}A \cap E = \varnothing$, where $E = B - (\bigcup_{n=1}^{\infty} T^{k_n}A \cap B)$. For (b), suppose $\mu(\bigcup_{i=1}^M T^{k_i}A) \leqq 1 - \varepsilon$. If $B = (\bigcup_{i=1}^M T^{k_i}A)^c$, then $\mu(B) \geqq \varepsilon$ and $T^{k_i}A \cap B = \varnothing$, $1 \leqq i \leqq M$, contradicting the definition of M. Thus here one can choose $N(A, \varepsilon) = M$ for uniformly sweeping out.

Uniform sweeping out on a sequence can be characterized by the conditions in Theorem (2.6) or Theorem (2.7)(b) restricted to the sequence.

3. Example

We will now construct a rank-1 transformation that is described in terms of concatenation of blocks as follows. The n-block is denoted by C_n, $n \geqq 1$, with $C_1 = (1)$. Let (r_n) and (m_n) be two sequences of positive integers exceeding 1. At the nth stage of the construction we have C_{n-1}. Let D_n consist of r_n copies of C_{n-1} with no spacer; hence

$$D_n = \underbrace{C_{n-1}C_{n-1}C_{n-1}\cdots C_{n-1}}_{r_n}.$$

Now C_n is formed from m_n copies of D_n, where the ith copy is followed by i spacers s; hence

$$C_n = D_n s D_n s s D_n \cdots D_n \underbrace{ss \cdots s}_{m_n}.$$

Let c_n and d_n be the lengths of C_n and D_n, respectively. We inductively define $r_n = n^4 c_{n-1}^2$ and $m_n = n^2 c_{n-1} + 1$, $n \geqq 2$. It easily follows that the corresponding measure preserving transformation T can be defined on the unit interval with Lebesgue measure μ and ergodicity follows from rank-1. The transformation T and the sequence (d_n) are related as in Theorem (3.1) below. It follows from (a) that T is weakly mixing.

THEOREM (3.1). (a) *T is uniformly mixing on (d_n).*

(b) *T is uniformly sweeping out on $([\alpha d_n])$ for all $\alpha \in (0, 1)$.*

(c) *For all rational $\alpha \in (0, 1)$ T is not mixing on $([\alpha d_n])$.*

PROOF. For (a), fix a positive integer u and let A and B be sets corresponding to specific sets of locations L_A and L_B, respectively, in the block C_u. Let $\varepsilon > 0$. Choose $V > u$ such that

$$\sum_{i=V}^{\infty} i^{-2} < \varepsilon. \tag{1}$$

The construction then implies that for $n \geqq V$, the spacer added after forming C_n is less than $\varepsilon\mu(E)$, where E is a set corresponding to a location in C_n.

Fix v and n with $V \leqq v < n$. Since $V > u$, L_A appears in the copies of C_u in D_{n-1}. Now C_{n-1} consists of m_{n-1} copies of D_{n-1} with additional spacer. Since $V \leqq v \leqq n-1$, $T^{d_v}A = A_1 \cup A_2$, where A_1 corresponds to locations L_A^* in C_{n-1} and $\mu(A_2) < \varepsilon\mu(A)$. Now D_n consists of r_n copies of C_{n-1} and each copy of C_{n-1} contains a copy of L_A^*.

We also have B corresponding to locations L_B^* in C_{n-1}, where L_B^* consists of copies of L_B. Hence each copy of C_{n-1} in D_n contains a copy of L_B^*.

Let E be a set corresponding to a location in D_n that is not in the initial segment of D_n of length m_n. The uniform distribution of the spacer lengths in the formation of C_n from D_n imply that $T^{d_n}E = E_1 \cup E_2$, where E_1 is uniformly distributed over the sets corresponding to locations in C_{n-1} and $\mu(E_2) < \varepsilon\mu(E)$.

Thus if B^* is the subset of B corresponding to locations in copies of L_B^* not in the initial segment of D_n of length m_n, then $T^{d_n}B^* = B_1 \cup B_2$, where B_1 is uniformly distributed over the sets corresponding to locations in C_{n-1} and $\mu(B_2) < \varepsilon\mu(B)$. The subset B_3 of B corresponding to locations in the initial segment of D_n of length m_n has measure $\mu(B_3) < \varepsilon\mu(B)$. Let μ_n be the measure corresponding to D_n. We therefore have

$$|\mu(T^{d_v}A \cap T^{d_n}B) - \mu(A)\mu(B)| \leqq |\mu(A_1 \cap T^{d_n}B^*) - \mu(A_1)\mu(B^*)| + 4\varepsilon$$
$$\leqq |\mu(A_1 \cap B_1) - \mu(A_1)\mu(B_1)| + 6\varepsilon$$
$$= |\mu(A_1)\mu(B_1)/\mu_n - \mu(A_1)\mu(B_1)| + 6\varepsilon$$
$$\leqq (1-\mu_n)/\mu_n + 6\varepsilon < 8\varepsilon.$$

Thus if A and B are sets corresponding to specific locations in C_u and $\varepsilon > 0$, then there exists V so that $V \leqq v < n$ implies

$$(2) \qquad |\mu(T^{d_v}A \cap T^{d_n}B) - \mu(A)\mu(B)| < 8\varepsilon.$$

In particular, we have shown that T is uniformly mixing on (d_n) for sets corresponding to locations in C_u for some u. Since measurable sets can be approximated by such sets, it follows that T is uniformly mixing on (d_n).

For (b), consider $0 < \alpha < 1$ and let $e_n = [\alpha d_n]$, $n \geqq 1$. We will show that T is uniformly sweeping out on (e_n). For each n, let t_n be the integer such that $\alpha/4 \leqq t_n/r_n < \alpha/4 + 1/r_n$. As in the proof of (a), let A be the set corresponding to specific locations L_A in a block C_u. Let $a_n = t_n\mu(A)/r_n$ and $a = \alpha\mu(A)/4$; hence $\lim_{n\to\infty} a_n = a$. Let A_n be the subset of A corresponding to the copies of L_A in the last t_n copies of C_{n-1} in D_n; hence $\mu(A) = a_n$, $n \geqq 1$. Let $B_n = T^{e_n}A_n$, $n \geqq 1$; hence $\lim_{n\to\infty} \mu(B_n) = a$.

Let $\eta > 0$. Choose M so that $(1-a)^M < \eta$. We can choose $V > u$ so large that if $V < i_1 < i_2 < \cdots < i_M$, then

$$(3) \qquad \left| \prod_{j=1}^{M} \mu(B_{i_j}^c) - (1-a)^M \right| < \eta.$$

We will show that V can also be chosen so large that

$$(4) \qquad \left| \mu\left(\bigcap_{j=1}^{M} B_{i_j}^c \right) - \prod_{j=1}^{M} \mu(B_{i_j}^c) \right| < \eta.$$

Let $\varepsilon = \eta/6M$. Choose V so large that (1) holds and $2/V^2 < \alpha$. Fix v and n such that $V < v < n$. Let E be the set corresponding to specific locations L_E in C_v. Therefore E corresponds to copies of L_E in the r_n copies of C_{n-1} in D_n.

Since $n > V$, $2/n^2 < \alpha$. It follows that $\alpha d_n/2 > m_n$; hence $B_n = T^{e_n}A_n = F_1 \cup F_2$, where F_1 is uniformly distributed over sets corresponding to locations in C_{n-1} and $\mu(F_2) < \varepsilon\mu(B_n)$. We therefore have

$$|\mu(E \cap B_n) - \mu(E)\mu(B_n)| \leqq |\mu(E \cap F_1) - \mu(E)\mu(F_1)| + 2\varepsilon$$
$$= |\mu(E)\mu(F_1)/\mu_n - \mu(E)\mu(F_1)| + 2\varepsilon$$
$$\leqq (1 - \mu_n)/\mu_n + 2\varepsilon < 4\varepsilon,$$

hence

$$|\mu(E \cap B_n) - \mu(E)\mu(B_n)| < 4\varepsilon. \tag{5}$$

Now (5) implies

$$|\mu(E \cap B_n^c) - \mu(E)\mu(B_n^c)| < 4\varepsilon. \tag{6}$$

We also have $E \cap B_n^c = E^* \cup G^*$, where E^* is a set corresponding to locations in C_n and $\mu(G^*) < \varepsilon$.

We now proceed by induction. We have $B_{i_1}^c = E_1 \cup G_1$, where E_1 is a set corresponding to locations in C_{i_1} and $\mu(G_1) < \varepsilon$. At the rth stage we have

$$\bigcap_{j=1}^{r} B_{i_j}^c = E_r \cup G_r, \tag{7}$$

where E_r is a set corresponding to locations in C_{i_r} and $\mu(G_r) < r\varepsilon$. We also have

$$\left| \mu\left(\bigcup_{j=1}^{r} B_{i_j}^c \right) - \prod_{j=1}^{r} \mu(B_{i_j}^c) \right| < 6r\varepsilon. \tag{8}$$

Let $v = i_r$, $n = i_{r+1}$, and $E = E_r$ in the analysis preceding (5) and (6). From (6), we obtain

$$|\mu(E_r \cap B_{i_{r+1}}^c) - \mu(E_r)\mu(B_{i_{r+1}}^c)| < 4\varepsilon. \tag{9}$$

We also have

$$E_r \cap B_{i_{r+1}}^c = E_{r+1}^* \cup G_{r+1}^*, \tag{10}$$

where E_{r+1}^* is a set corresponding to locations in $C_{i_{r+1}}$ and $\mu(G_{r+1}^*) < \varepsilon$. Therefore (10) implies

$$(E_r \cup G_r) \cap B_{i_{r+1}}^c = E_{r+1}^* \cup G_{r+1}^* \cup (G_r \cap B_{i_{r+1}}^c). \tag{11}$$

Let $E_{r+1} = E_{r+1}^*$ and $G_{r+1} = G_{r+1}^* \cup (G_r \cap B_{i_{r+1}}^c)$; hence (7) and (11) imply (7) holds with r replaced by $r + 1$ with $\mu(G_{r+1}) < (r + 1)\varepsilon$. From (7), (8), and (9) we obtain (8) with r replaced by $r + 1$. This completes the induction step.

With $r = M$ in (8) and $\varepsilon = \eta/6M$, we obtain (4). Since $(1 - a)^M < \eta$, (3) and

(4) imply $\mu(\bigcup_{j=1}^{M} B_{i_j}) > 1 - 3\eta$. Thus the union of any M iterates of A on (e_n), $n > V$, will have measure greater than $1 - 3\eta$.

In general, let $\mu(B) > 0$ and $\eta > 0$. Choose M so that $(1 - \alpha\mu(B)/8)^M < \eta$. There exists $\delta > 0$ such that $\mu(A \triangle B) < \delta$ implies $\mu(A) < \mu(B)/2$ and

$$\left| \mu\left(\bigcup_{j=1}^{M} T^{i_j}A \right) - \mu\left(\bigcup_{j=1}^{M} T^{i_j}B \right) \right| < \eta, \tag{12}$$

for any increasing i_j, $1 \leqq j \leqq M$.

There exists a block C_u and a set A corresponding to locations in C_u such that $\mu(A \triangle B) < \delta$; hence $\mu(A) > \mu(B)/2$. With $a = \alpha\mu(A)/4$, it follows that $(1 - a)^M < \eta$. Let V be as above. By (12) we conclude that any M iterates of B on (e_n), $n > V$, has measure greater than $1 - 4\eta$. Thus any $M + V$ iterates of B on (e_n) has measure greater than $1 - 4\eta$. This proves T is uniformly sweeping out on (e_n).

For (c), consider rational $\alpha = a/b$. Choose v such that $c_v > 3b$ and let A be the set corresponding to the initial c_v copies of C_v in D_{v+1}; hence $\mu(A) < c_v/(v+1)^4 c_v^2 < 1/c_v < 1/3b$. Now $d_n = n^4 c_{n-1}^3$ and we can choose $n > v$ of the form $n = kb$. It follows that $e_n = ad_n/b$ is a multiple of c_{n-1}. Since $b - a \geqq 1$, at least the initial $1/b$th copies of C_{n+1} in D_n coincide with copies of C_{n-1} when translated by e_n. The copies of C_v in the initial $1/b$th copies of C_{n-1} in D_n therefore coincide with copies of C_v in copies of C_{n-1} when translated by e_n. It follows that $\mu(T^{e_n}A \cap A) > \mu(A)/2b$. Since $\mu(A) < 1/3b$ and $e_n \to \infty$ as $n = kb \to \infty$, T cannot be mixing on (e_n).

We note that it is also possible to construct irrational α that is well-approximated by rationals such that T is not mixing on (e_n). However, we do not know if there exists irrational α such that T is mixing on (e_n).

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